

Arzelà-Ascoli theorem via Wallman compactification

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Abstract

In the paper, we recall the Wallman compactification of a Tychonoff space T (denoted by $\text{Wall}(T)$) and the contribution made by Gillman and Jerison. Motivated by the Gelfand-Naimark theorem, we investigate the homeomorphism between $C^b(T)$ and $C(\text{Wall}(T))$. Along the way, we attempt to justify the advantages of Wallman compactification over other manifestations of Stone-Čech compactification. The main result of the paper is a new form of Arzelà-Ascoli theorem, which introduces the concept of equicontinuity along ω -ultrafilters.

Keywords : Arzelà-Ascoli theorem, Wallman compactification, Stone-Čech compactification, ultrafilters

1 Introduction

The classical Arzelà-Ascoli theorem ([5] on page 278) plays an important role in functional analysis. It characterizes the relatively compact subsets of the space of complex-valued continuous functions $C(X)$, with X a compact space, as those which are equibounded and equicontinuous. The space $C(X)$ is given the standard norm

$$\|f\| := \sup_{x \in X} |f(x)|$$

The theorem admits numerous generalizations. A version for locally compact space X and metric space Y can be found in [5] on page 290. The topology on $C(X, Y)$ is the topology of uniform convergence on compacta.

In [6], Bogdan Przeradzki studied the existence of bounded solutions to the equations $x' = A(t)x + r(x, t)$, where A is a continuous function taking values in the space of bounded linear operators in a Hilbert space and r is a nonlinear continuous mapping. He came up with a characterization of relatively compact subsets of the space of bounded and continuous functions $C^b(\mathbb{R}, E)$, where E is a Banach space. In addition to pointwise relative compactness and equicontinuity, the following condition was introduced:

(P) For any $\varepsilon > 0$, there exist $T > 0$ and $\delta > 0$ such that if $\|x(T) - y(T)\| \leq \delta$ then $\|x(t) - y(t)\| \leq \varepsilon$ for $t \geq T$ and if $\|x(-T) - y(-T)\| \leq \delta$ then $\|x(t) - y(t)\| \leq \varepsilon$ for $t \leq -T$ where x and y are arbitrary functions in \mathcal{F} .

This idea was further studied by Robert Stańczy in [8], while investigating the existence of solutions to Hammerstein equations in the space of bounded and continuous functions. The results were applied to Wiener-Hopf equations and to ODE's. In [4] the author, together with Bogdan Przeradzki recast (P) in a setting where X is σ -locally compact Hausdorff space and Y is a complete metric space. In this paper, we recall the obtained result (as theorem 7) without proof.

Bogdan Przeradzki suggested that Arzelà-Ascoli theorem could be described in terms of ultrafilters. The motivation comes from Gelfand-Naimark theorem in [2], which states that $C^b(T)$ and $C(\beta T)$ are $*$ -isomorphic as C^* -algebras (so in particular homeomorphic), where βT is the Stone-Čech compactification. The idea is to write the classical Arzelà-Ascoli theorem for $C(\beta T)$ and to interpret it from the perspective of $C^b(T)$. To this end, we will need an explicit form of the homeomorphism between these spaces.

We will use Wallman ultrafilter construction rather than the original approach by Stone or Čech. The reason is that Wallman topology will be more convenient to work with than the weak* topology of βT . We attempt to explain the advantages of this topology at the beginning of section 2. Moreover, Wallman's original construction in [11] has been improved by Gillman and Jerison in [3] and even further by Frink in [1] or Steiner in [9]. Although, we will not need full generality, the crucial parts of the construction are briefly summarized in this section.

The main results are described in section 3. Theorem 4 describes the homeomorphism between $C^b(T)$ and $C(\text{Wall}(T))$, as anticipated by the Gelfand-Naimark theorem. The culminating point is theorem 6, which characterizes relatively compact subsets of $C^b(T)$ in terms of ω -ultrafilters. The theorem introduces the concept of equicontinuity along ω -ultrafilters, which accounts for classical equicontinuity and $C^b(X, Y)$ -extension property in theorem 7.

2 Wallman compactification

Throughout the whole paper, we will assume that (T, τ_T) is a *Tychonoff space* i.e. it is T_1 and completely separates points from closed sets (there exists a function f which is 0 on the given point and 1 on the given closed set). Moreover, we denote

$$\mathcal{Z}(T) := \left\{ f^{-1}(0) : f \in C^b(T) \right\}$$

The elements of $\mathcal{Z}(T)$ are called the *zero sets*. Observe that $\mathcal{Z}(T)$ is closed under finite intersections. To the best of our knowledge, the importance of zero sets in the context of Wallman compactification were first noticed by Gillman and Jerison in [3], particularly chapter 6. The essence of $\mathcal{Z}(T)$ is grasped by the following lemma, which appears (as a part of a proof) in [10] on page 23.

Lemma 1. *For any $A, B \in \mathcal{Z}(T)$ such that $A \cap B = \emptyset$ there are $U, V \in \tau_T$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Intuitively, zero sets provide a substitute for the separation axiom T_4 (normality). On the basis of these sets, Gillman and Jerison built the Wallman compactification, which was originally carried out using all closed subsets of T (comp. [11]).

A family $\mathcal{U} \subset \mathcal{Z}(T)$ is called a *Wallman ultrafilter* or ω -*ultrafilter* if

($\omega 1$) Any finite intersection of elements in \mathcal{U} is nonempty.

($\omega 2$) The family \mathcal{U} is maximal.

We will denote the set of all ω -ultrafilters on T by $\text{Wall}(T)$. It is easy to observe that every point $t \in T$ determines a *principal ω -ultrafilter*

$$\mathcal{P}_t := \left\{ f^{-1}(0) \in \mathcal{Z}(T) : f(t) = 0 \right\}$$

The function $\wp : T \rightarrow \text{Wall}(T)$ given by

$$\forall t \in T \quad \wp(t) := \mathcal{P}_t$$

is called a *principal function*.

The first step in introducing the topology on $\text{Wall}(T)$ is defining the operator $*$: $\tau \rightarrow 2^{\text{Wall}(T)}$ with

$$\forall U \in \tau_T \quad U^* := \left\{ \mathcal{U} \in \text{Wall}(T) : T \setminus U \notin \mathcal{U} \right\}$$

It is a trivial observation, yet still useful, that

$$\forall U \in \tau_T \quad t \in U \iff \wp(t) \in U^* \tag{1}$$

Among the other properties of $*$ -operator, we recall that if $U, V \in \tau_T$ then $(U \cap V)^* = U^* \cap V^*$ and $(U \cup V)^* = U^* \cup V^*$. Furthermore, if $U \subset V$ then $U^* \subset V^*$, so the operator $*$ is increasing with respect to inclusion. We may conclude that the family of sets U^* where $U \in \tau_T$, is a topological base. The topology, which is thus introduced on $\text{Wall}(T)$, is called *Wallman topology* and we denote it by τ^* .

Theorem 2. (*Wallman compactification*)

The pair $(\text{Wall}(T), \wp)$ is the Stone-Ćech compactification.

We omit the proof, sketching only 2 of its aspects. First, in order to establish that $(\text{Wall}(T), \tau^*)$ is Hausdorff, we use lemma 1. This part heavily relies on zero sets. Originally, the construction was done for normal space T by Wallman in [11]. It was Gillman and Jerison, who realized the importance of zero sets thus requiring of T only to be Tychonoff.

The proof usually goes on to show that $(\text{Wall}(T), \tau^*)$ is compact, and that \wp is a homeomorphism of T and a dense subspace of $\text{Wall}(T)$. At the final stage, in order to prove that $(\text{Wall}(T), \wp)$ is the Stone-Čech compactification, we can simply verify that every $f \in C^b(T)$ extends to $\hat{f} \in C(\text{Wall}(T))$ in the sense $\hat{f} \circ \wp = f$. This characterization (among many others) can be found in [10] on page 25.

For every $\mathcal{U} \in \text{Wall}(T)$ we consider

$$\mathcal{U}_f := \left\{ A \subset \overline{\text{Im}(f)} : f^{-1}(A) \in \mathcal{U} \right\} \quad (2)$$

It can be shown that the intersection $\bigcap \mathcal{U}_f$ is not only nonempty, but moreover consists of exactly one point. This point is called the *limit of the function along ω -ultrafilter* and denoted by $\lim_{\mathcal{U}} f$. Last but not least, we can define $\hat{f} : \text{Wall}(T) \rightarrow \mathbb{R}$ by

$$\forall \mathcal{U} \in \text{Wall}(T) \quad \hat{f}(\mathcal{U}) := \lim_{\mathcal{U}} f \in \bigcap \mathcal{U}_f \quad (3)$$

The proof ends with the verification of the continuity of \hat{f} and the identity $\hat{f} \circ \wp = f$.

3 Arzelà-Ascoli theorem

The current section will prove that $C^b(T)$ and $C(\text{Wall}(T))$ are homeomorphic. This does not come as a surprise if one recalls the famous Gelfand-Naimark theorem, which can be found in [2]. The Stone-Čech compactification can manifest itself in a variety of forms:

- βT , as an embedding of T into a compact space $[0, 1]^{C(T, [0, 1])}$, or $\mathbb{R}^{C^b(T, \mathbb{R})}$.
- $\text{Wall}(T)$, as a family of ω -ultrafilters (precisely the construction presented above).
- $\Delta(C^b(T))$, as a family of nonzero algebra homomorphisms $\chi : C^b(T) \rightarrow \mathbb{C}$, called *characters*.
- $\mathcal{I}(C^b(T))$, as maximal ideals of algebra $C^b(T)$.

The last two approaches were comprehensively studied in [7] in chapter 14. The advantage of $\text{Wall}(T)$ over other forms of the Stone-Čech compactification is the simplicity of open sets. While $\beta T, \Delta(C^b(T))$ and $\mathcal{I}(C^b(T))$ all use some sort of weak* topology, Wallman compactification enjoys a very pleasant Wallman topology (described above), which is a bit easier to handle.

Before focusing on the homeomorphism, we prove a vital property of the limit along ultrafilter.

Lemma 3. *If $f \in C^b(T)$ then for every $\varepsilon > 0$ we have*

$$T \setminus \left\{ t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \right\} \notin \mathcal{U} \quad (4)$$

Proof

Suppose that (4) does not hold for some $\varepsilon > 0$, i.e.

$$\left\{ t \in T : |f(t) - \lim_{\mathcal{U}} f| \geq \varepsilon \right\} \in \mathcal{U} \quad (5)$$

We put $A := \mathbb{C} \setminus B(\lim_{\mathcal{U}} f, \varepsilon)$, which is obviously a closed set such that $\lim_{\mathcal{U}} f \notin A$. Moreover, (5) means that

$$f^{-1}(A) \in \mathcal{U} \xrightarrow{(2)} A \in \mathcal{U}_f \xrightarrow{(3)} \lim_{\mathcal{U}} f \in A$$

which is a contradiction. ■

Theorem 4. *The function $\Gamma : C^b(T) \rightarrow C(\text{Wall}(T))$ given by*

$$\forall_{f \in C^b(T)} \Gamma(f) := \hat{f} \quad (6)$$

is a homeomorphism.

Proof

At first, we prove the continuity of Γ . Fix $\mathcal{U} \in \text{Wall}(T)$, $\varepsilon > 0$ and suppose that $d_{C^b(T)}(f, g) < \varepsilon$. By lemma 3 we have

$$T \setminus \left\{ t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \right\} \notin \mathcal{U} \quad \text{and} \quad T \setminus \left\{ t \in T : |g(t) - \lim_{\mathcal{U}} g| < \varepsilon \right\} \notin \mathcal{U}$$

which means

$$\begin{aligned} \mathcal{U} &\in \left\{ t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \right\}^* \cap \left\{ t \in T : |g(t) - \lim_{\mathcal{U}} g| < \varepsilon \right\}^* \\ &= \mathcal{U} \in \left\{ t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \wedge |g(t) - \lim_{\mathcal{U}} g| < \varepsilon \right\}^* \end{aligned}$$

Consequently, we obtain

$$\mathcal{U} \in \left\{ t \in T : |f(t) - g(t) - (\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g)| < 2\varepsilon \right\}^*$$

and conclude that

$$\mathcal{U} \in \left\{ t \in T : |\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g| < 2\varepsilon + |f(t) - g(t)| \right\}^* \implies \mathcal{U} \in \left\{ t \in T : |\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g| < 3\varepsilon \right\}^*$$

Finally, the set $\left\{ t \in T : |\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g| < 3\varepsilon \right\}$ cannot be empty, since $\mathcal{U} \notin \emptyset^*$.

Hence, $|\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g| < 3\varepsilon$ and since the choice of \mathcal{U} was arbitrary, we establish that $d_{C(\text{Wall}(T))}(\hat{f}, \hat{g}) < 3\varepsilon$, proving continuity of \hat{f} .

For surjectivity, let $F \in C(\text{Wall}(T))$. If we set $f = F \circ \wp$ then due to the fact that $\wp : T \rightarrow \wp(T)$ is a homeomorphism, we have $f \in C^b(T)$. Both functions $\Gamma(f)$ and F agree on a dense set $\wp(T)$, hence by continuity they are equal $\text{Wall}(T)$.

In order to prove that Γ is an injection, suppose that $\Gamma(f) = \Gamma(g)$. This means that $\lim_{\mathcal{U}} f = \lim_{\mathcal{U}} g$ for every $\mathcal{U} \in \text{Wall}(T)$. Focusing on the principal ω -ultrafilters, we immediately conclude that $f \equiv g$.

It remains to prove the continuity of Γ^{-1} . If we suppose that

$$d_{C(\text{Wall}(T))}(\Gamma(f), \Gamma(g)) \leq \varepsilon \iff \forall \mathcal{U} \in \text{Wall}(T) \quad |\lim_{\mathcal{U}} f - \lim_{\mathcal{U}} g| \leq \varepsilon$$

It suffices to put $\mathcal{U} = \mathcal{P}_t$ where $t \in T$ in order to obtain $d_{C^b(T)}(f, g) \leq \varepsilon$, which ends the proof. ■

The function Γ defined in (6) is actually a well-know Gelfand transform. As mentioned before, the celebrated Gelfand-Naimark theorem states that Γ is in fact a $*$ -isometry between $C^b(T)$ and $C(\text{Wall}(T))$. However, we need not resort to such heavy machinery. The knowledge that Γ is 'merely' a homeomorphism and the corollary below is sufficient for all our considerations.

Corollary 5. *If the set $\mathcal{F} \subset C^b(T)$ is relatively compact then $\hat{\mathcal{F}} := \Gamma(\mathcal{F})$ is relatively compact and vice versa.*

We are ready to state and prove the culminating theorem of this paper.

Theorem 6. *(Arzelà-Ascoli via Wallman compactification)
A family $\mathcal{F} \subset C^b(T)$ is relatively compact if and only if*

(AA1) \mathcal{F} is pointwise bounded, i.e. the set $\{f(t) : f \in \mathcal{F}\}$ is bounded for every $t \in T$

(AA2) \mathcal{F} is ω -equicontinuous, i.e.

$$\forall \mathcal{U} \in \text{Wall}(T) \quad \exists_{\varepsilon > 0} \quad \exists_{V \in \tau_T} \quad \forall_{f \in \mathcal{F}} \quad \forall_{t \in V} \quad |f(t) - \lim_{\mathcal{U}} f| < \varepsilon$$

Proof

Suppose that $\mathcal{F} \subset C^b(T)$ is relatively compact. By corollary 5, this is equivalent to the relative compactness of $\hat{\mathcal{F}}$. By classical Arzelà-Ascoli theorem, we have that

(AA ω 1) $\hat{\mathcal{F}}$ is pointwise bounded, i.e. the set $\{\hat{f}(\mathcal{U}) : f \in \mathcal{F}\}$ is bounded for every $\mathcal{U} \in \text{Wall}(T)$

(AA ω 2) $\hat{\mathcal{F}}$ is equicontinuous, i.e.

$$\forall_{\substack{\mathcal{U} \in \text{Wall}(T) \\ \varepsilon > 0}} \exists_{\substack{V \in \mathcal{T}_T \\ \mathcal{U} \in V^*}} \forall_{\substack{f \in \mathcal{F} \\ \mathcal{V} \in V^*}} |\hat{f}(\mathcal{V}) - \hat{f}(\mathcal{U})| < \varepsilon$$

At first we observe that pointwise boundedness at every point, in view of equicontinuity, is equivalent to pointwise boundedness on a dense set $\wp(T)$. Thus the condition **(AA ω 1)** can be replaced with **(AA1)**.

Observe that the implication

$$\forall_{\mathcal{V} \in V^*} |\lim_{\mathcal{V}} f - \lim_{\mathcal{U}} f| < \varepsilon \implies \forall_{t \in V} |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \quad (7)$$

holds. Indeed, if $t \in V$ then by (1) we know that $\mathcal{P}_t \in V^*$. Since $\lim_{\mathcal{P}_t} f = f(t)$, we conclude that (7) holds and thus **(AA ω 2)** implies **(AA2)**.

We aim to show that

$$\forall_{t \in V} |f(t) - \lim_{\mathcal{U}} f| < \varepsilon \implies \forall_{\mathcal{V} \in V^*} |\lim_{\mathcal{V}} f - \lim_{\mathcal{U}} f| \leq \varepsilon \quad (8)$$

which will prove that **(AA2)** implies **(AA ω 2)**. We take $\mathcal{V} \in V^*$ and assume that $V \subset \{t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon\}$, which implies

$$\mathcal{V} \in \left\{t \in T : |f(t) - \lim_{\mathcal{U}} f| < \varepsilon\right\}^* \quad (9)$$

Moreover, by lemma 3 we have

$$T \setminus \left\{t \in T : |f(t) - \lim_{\mathcal{V}} f| < \eta\right\} \notin \mathcal{V} \iff \mathcal{V} \in \left\{t \in T : |f(t) - \lim_{\mathcal{V}} f| < \eta\right\}^* \quad (10)$$

for every $\eta > 0$. Intersecting (9) and (10) we obtain

$$\mathcal{V} \in \left\{t \in T : |\lim_{\mathcal{U}} f - \lim_{\mathcal{V}} f| < \varepsilon + \eta\right\}^*$$

for every $\eta > 0$. Reasoning as before, the set $\{t \in T : |\lim_{\mathcal{U}} f - \lim_{\mathcal{V}} f| < \varepsilon + \eta\}$ cannot be empty, since $\mathcal{V} \notin \emptyset^*$. Since the choice of η was arbitrary, we conclude that $|\lim_{\mathcal{U}} f - \lim_{\mathcal{V}} f| \leq \varepsilon$. We proved (8) and consequently, **(AA ω 2)** is equivalent to **(AA2)**. ■

As a final note, let us compare the obtained result with the previous work of the author together with Bogdan Przeradzki. In [4], the following generalization of Arzelà-Ascoli theorem was proved:

Theorem 7. (*Arzelà-Ascoli for σ -locally compact Hausdorff space*)

Let (X, τ_X) be σ -locally compact Hausdorff space and (Y, d_Y) be a metric space. The set $\mathcal{F} \subset C^b(X, Y)$ is relatively compact iff

(KP1) \mathcal{F} is pointwise relatively compact

(KP2) \mathcal{F} is equicontinuous

(KP3) \mathcal{F} satisfies $C^b(X, Y)$ -extension property, i.e.

$$\forall \varepsilon > 0 \quad \exists_{D \in X} \forall_{f, g \in \mathcal{F}} d_{C^b(D, Y)}(f, g) < \delta \implies d_{C^b(X, Y)}(f, g) < \varepsilon$$

where $D \in X$ means that D is a compact subset.

In our considerations, we relaxed the assumptions on X to being simply a Tychonoff space and took $Y = \mathbb{C}$. We observe that **(AA2)** corresponds to **(KP2)** when we consider only principal ω -ultrafilters. Indeed, we have

$$\begin{aligned} & \forall_{t_* \in T} \exists_{\varepsilon > 0} \exists_{V \in \tau_T} \forall_{f \in \mathcal{F}} \forall_{t \in V} |f(t) - \lim_{\mathcal{P}_{t_*}} f| < \varepsilon \\ \iff & \forall_{t_* \in T} \exists_{V_{t_*} \in \tau_T} \forall_{f \in \mathcal{F}} \forall_{t \in V_{t_*}} |f(t) - f(t_*)| < \varepsilon \end{aligned}$$

which is exactly **(KP2)**. Consequently, the rest of ω -ultrafilters in $\text{Wall}(T)$, sometimes referred to as *free ω -ultrafilters*, play the same role as **(KP3)**.

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